From Communication Equilibria to Correlated Equilibria*

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Abstract

We show that the set of communication equilibria of any finite Bayesian game can be implemented in correlated equilibria of an extended game, where before playing in the underlying game players engage in a possibly infinitely long, plain, pre-play communication.

Keywords: Bayesian games, communication equilibria, correlated equilibria, pre-play communication

JEL Classification Numbers:

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1 Introduction

In Crawford and Sobel (1982) an informed player has the possibility to partially reveal his private information by sending a message to another player who has to choose an action. There are Bayesian equilibrium payoffs of this game that Pareto improve the equilibrium of the " silent " game where the informed player cannot communicate. Imagine that the players can communicate through a mediator in this situations. The informed player sends information about his type to the mediator and not to the receiver. The mediator selects an action from a distribution which depends on the sender's declaration. The mediator suggests the selected action to the receiver. In this manner the players may improve even upon the unmediated extension. Mitusch and Strausz (2000) give an intuitive[1] example demonstrating this improvement. In games with incomplete information the largest set of non-cooperative solutions achievable when arbitrary means of communication are available, before choosing actions, is the set of communication equilibria (Myerson 1982, Forges 1986).

Aumann and Hart (2003) characterize the set of Bayesian Nash equilibrium payoffs of 2 players finite games with one-sided[2] incomplete information extended by an infinitely long plain (i.e. unmediated) communication phase. Denote the extended game of Aumann and Hart by $EXT\Gamma$, where Γ is the underlying " silent " game. They show that multistage, possibly infinitely long communication can do better than just a single or finitely many stages of communication. Nevertheless, even if the communication is infinitely long, there is still place for improvement.

In our Theorem we show, that players can achieve any payoff from the set of communication equilibrium payoffs of the underlying game in Bayes Nash equilibria of $EXT\Gamma$ if players' communication strategies can be correlated.

Following Harsányi (1967) a Bayesian game $\Gamma =< I, L^i, A^i, g^i, \lambda, i \in I >$ is described by a finite set of players I, the finite set of possible types L^i of player i, a compact set of actions A^i for player i. Let $L = \Pi_{i=1}^I L^i$ and $A = \Pi_{i=1}^I A^i$ and $g^i : L \times A \to \mathbb{R}$ the payoff function of player i and $\lambda \in \Delta L$ is the players common prior over L.

Suppose, players participate in an extended game Γ^q which is as follows: Players learn their types as in Γ. Players can send information about their types privately to a mediator. Suppose, the sent message profile is l. The mediator randomizes

[1]The intuition behind this result is that the mediated equilibrium is not constrained by the incentives of a type of the informed player to be indifferent between sending different messages. This happens when the informed player reveals his information partially and mixes between messages.

[2]See Amitai (1996) for incomplete information on both side.

over the action profiles according to a function $q : L \to \Delta A$ of the players' messages. Suppose, the result of the randomization is $a = (a^1, \ldots, a^I)$. The mediator then sends private messages to the players which can be interpreted as the players' suggested action. That is player i privately receives $a^i \in A^i$. Finally players choose actions. If it is a Bayesian Nash equilibrium of the extended game that given q the players are sincere about their types and obediently follow the suggestion of the mediator then q is a canonical communication equilibrium. Let $ME(\Gamma)$ denote the set of canonical communication equilibrium payoffs of a game Γ with incomplete information. Let $BE(\Gamma)$ be the set of Bayesian Nash equilibrium payoffs of some Γ. Then $\cup_q BE(\Gamma^q) = ME(\Gamma)$. According to the revelation principle, these direct (i.e. canonical) mechanisms cover all possibly more complicated communication procedures[3]. More precisely, for all $c : \Pi_{i\in I} M^i \to \Delta(\Pi_{i\in I} H^i)$ non-canonical communication device ($M^i \neq L^i, H^i \neq A^i$) $BE(\Gamma^c) \subseteq ME(\Gamma)$.

Our main question is whether the players can achieve all possible outcomes in $ME(\Gamma)$ in an incentive compatible manner with a help of a mediator who cannot condition his randomization on received messages from the players?

I show that the answer is positive if we assume that players can have a possibly infinitely long unmediated interim conversation after receiving the mediators recommendation about how to communicate. In the Crawford and Sobel model or in the model of Aumann and Hart it means that players can achieve anything which is possible with canonical communication devices by engaging in a longer two-sided communication (EXT) if players' communication strategies can be correlated. That is, it is sufficient to achieve efficiency in any game if players can use correlated communication strategies[4].

We apply Aumann's *strategic form* correlated equilibrium for games with incomplete information to answer this question formally. *Any* incomplete information game can be extended with a correlation device μ which randomizes over the set of *strategy profiles* $s = (s^1, \ldots, s^I) \in \Pi_{i\in I} {A^i}^{L^i}$ of the underlying game. Denote the following extended game by Γ^μ. Players learn their types as in Γ. μ randomizes and informs player i privately about s^i, the ith coordinate of the realization. Finally, players choose strategies and play in Γ. μ is a correlated equilibrium distribution of Γ if in Bayes Nash equilibrium of the extended game Γ^μ, player i of type l^i receiving recommendation s^i, obediently plays according to $s^i(l^i)$. Denote the set of correlated equilibrium payoffs of Γ by $CE(\Gamma) = \cup_\mu BE(\Gamma^\mu)$. It

[3]See Gerardi and Myerson (2005), where the revelation principle fails due to imposing sequential rationality.

[4]See Dimitri (2000) how correlated errors in Rubinstein's (1989) Electronic Mail Game can enhance the possibility of efficient coordination and outcomes. Other real world justification for correlated, interdependent communication stem from social, cultural backgrounds; the pragmatic use of language; the effect of media etc..

is obvious that $BE(\Gamma) \subseteq CE(\Gamma) \subseteq ME(\Gamma)$. Forges (1990) was interested in finding a universal unmediated interim cheap talk extension of Γ, $ext\Gamma$ for which $ME(\Gamma) = CE(ext\Gamma)$.

Forges (1990): *For any finite game Γ with at least 3 players, every payoff in $ME(\Gamma)$ is a* correlated equilibrium *payoff of a universally extended game $ext\Gamma$, where after having received their information as in Γ, players talk for* two stages *by sending public messages. In case of 3 players the message spaces have to have the size of a continuum.*

Forges's message is that a mediator does not need information about the players' type and randomize accordingly to achieve all the possible communication equilibrium payoffs of Γ. If players can have interim communication after the correlation phase, it is enough if the mediator works as a correlation device for the extended game $ext\Gamma$. Notice, that the correlation device does not give suggestions about how to choose actions in Γ, but how to communicate in $ext\Gamma$ and choose actions in Γ according to the history of the communication.

Our main contribution is the extension of Forges (1990) for general finite games with countable message spaces even with 2 or 3 players. Another way to look at our contribution is the calculation of the set of correlated equilibria of $EXT\Gamma$ suggested by Aumann and Hart (2003).

Theorem: *For any finite Bayesian game Γ, almost every payoff in $ME(\Gamma)$ is a* correlated equilibrium *payoff of a universally extended game $EXT\Gamma$, where after having received their information as in Γ, players can send messages from countable message spaces to each other simultaneously in discrete time, possibly for* infinitely many stages. *The communication terminates in finite time with probability 1.*

The Theorem basically states that the set of correlated equilibrium payoffs of the extended game of Aumann and Hart (2003) is the set of communication equilibrium payoffs of the underlying game. In short:

$$ME(\Gamma) \approx CE(EXT\Gamma).$$

Our correlated communication strategies are constructed in a way that players may find profitable deviations from the recommended strategy in the absence of punishment. These deviations cannot be detected with probability 1, but arbitrarily close to it. The deviator is then punished on his interim individually rational level. For this reason with our construction players can only achieve strictly interim individually rational payoffs from the set of communication equilibrium payoffs in correlated equilibria of the extended game. However, if players are allowed to communicate in continuous time, deviators can find profitable deviations from the suggested strategy with positive probability only if these deviations are detected with probability 1. Thus, we have the full implementation of $ME(\Gamma)$ in correlated equilibria of $EXT_{continuous}\Gamma$.

Aumann and Hart (2003) have no reason to care about punishments in their equilibria. In Forges (1990) there is no need of deterring deviations as long as the number of players is more than 3 or players can use messages from an interval if the number of players is 3. This is because in these cases one can design correlated communication strategies, where unilateral deviations cannot affect the induced distribution, the honest players' information and the deviator cannot learn more about the others' types and actions. To put it simply, the deviator finds herself in a situation just as in Γ^q after any deviation. However, in case of 2 players it is obviously not the case as it is demonstrated in Forges's (1990) example 2.4. Players can affect the induced distribution and the information of the opponent. This is because players have to signal and learn their actions simultaneously. As Forges (1990) points out, the main problem comes from the combination of signalling and decision. Forges (1985) gives solutions for special cases when this problem does not arise.

Our novel construction of correlated strategies opens the scope of interim punishments which deter players from deviation. That is, the mediator can give suggestions which the players would not obey in the absence of punishments. The issue of punishment is more delicate then it seems for the first sight. The equilibrium hinges on the possibility of infinitely long communication. To punish effectively, players cannot know in advance the exact time when the communication terminates and the time they learn their actions.

Ben-Porath (2003) uses dominated Bayes Nash equilibria of Γ to deter deviations during the communication (and to maintain sequential rationality). However, in his construction, a player who deviates only in the last stage of communication learns his action and updates his prior, thus the original dominated Bayes Nash equilibrium cannot apply as punishment.

Unlike in Aumann and Hart, in our equilibria players do not update their priors in each (even) stage of the communication, but it happens at an *uncertain stage* which is chosen by the correlation device. Knowing the stage or having a good inference when updating happens, a player could have incentive to deviate in that stage of communication, learn his action, update his prior and keep his opponent from updating by sending an incorrect message. The conditions for communication equilibria do not guarantee that players could not benefit from such deviations. The other player may detect this deviation but punishing a player with updated priors is impossible in general without stringent restrictions on the payoffs.

Our construction resolves this problem and applies to Forges (1990) with 2 or 3 players with countable message spaces and fixes[5] the problem of punishment in Ben-Porath (2003). Finally, it shows that players can achieve any payoff in $ME(\Gamma)$ in the game of Aumann and Hart (2003) if the communication strategies can be correlated.

In Section 2 we give the definitions we need and state the Theorem. Section 3 gives the intuitive idea of the proof. It consists of 3 parts. First we show the existence of correlated communication strategies with which players can mimic any communication device. More precisely, for any q there are correlated random communication strategies and decision rules such that if players follow the recommended communication strategies associated to their types, then given the history of communication it is a mutual best reply to play according to the decision rules and achieve the same payoffs as offered by q whenever q is a communication equilibrium of Γ. We stress that following the recommended communication strategy is not necessarily a best reply given players' information. Second we require sufficient conditions from the mechanism, called weak security, such that deviations from the recommendations can be detected and punished. Detecting deviations can be solved easily. However, as we pointed out it is important that the deviator should be caught before he learns his action. Finally we show how to embed the mimicking strategies into weakly secure ones in a way that a detected deviator can be punished with high probability. The exact, constructive proof can be found in the Appendix.

2 Preliminaries and the Theorem

First we recall the basic definitions of signalling function, Bayesian Nash equilibrium, correlated equilibrium, communication equilibrium of a Bayesian game and the notion of interim individually rational payoffs.

Let Γ be a Bayesian game defined above with finite action sets and $S^i = \{s^i | s^i : L^i \to A^i\}$ the set of pure strategies of player i in Γ and identify I with $\{1, \ldots, I\}$. Given a compact set E denote ΔE the set of Borel probability measures over the set E and $\operatorname{supp} \mu = \{e \in E | \forall U : U \textit{ open and } e \in U, \mu(U) > 0\}$ for $\mu \in \Delta E$ is the support of μ. If E^i is compact for all $i \in I$ we use the product topology on $E = \Pi_{i \in I} E^i$. We extend linearly g^i to $\rho \in \Delta A$ as $g^i(\rho, l) = \mathbb{E}_\rho g^i(a, l)$.

We introduce now the general notion of a signalling function c which receives private inputs from the players and produces private outputs for the players as

[5] In our games players are not allowed to use any "hard device" such as urns, envelopes as in Krishna (2004) or recording machine as in the corrigendum of Ben-Porath (2003).

possibly random functions of the inputs. Let $c : \Pi_{i\in I} M_c^i \to \Delta H_c$ a Borel function, where M_c^i, H_c^i are compact sets.

1. players send a private message $m_c^i \in M_c^i$, possibly chosen randomly according to $\sigma_c^i \in \Delta M_c^i$ to the signaling function c as private inputs,

2. c selects $h_c \in H_c = \Pi_{i\in I} H_c^i$ according to $c(m_c) \in \Delta H_c$

3. player i is told privately the ith coordinate h_c^i of the realization of $c(m_c)$

Let $\sigma_c = (\sigma_c^1, \dots, \sigma_c^I)$ and denote by $c(\sigma_c) \in \Delta H_c$ the distribution induced by c and any, possibly correlated mixed messages $\sigma_c \in \Delta M_c$. Consider the following extended game Γ^c:

1. player i learns his type $l^i \in L^i$ for all $i \in I$ as in Γ

2. players send private messages $m_c = (m_c^1, \dots, m_c^I)$ to c

3. players receive private messages h_c^i from c, where $h_c = (h_c^1, \dots, h_c^I)$ is chosen according to $c(m_c)$

4. players choose actions in Γ

A strategy in the extended game Γ^c is called a $c-$protocol. Formally:

Definition 1 *A $c-$protocol $(\sigma_c, \rho_c)(.)$ consists of the communication strategies $\sigma_c^i(.) : L^i \to \Delta M_c^i$ and decision rules that are Borel functions $\rho_c^i : L^i \times H_c^i \to \Delta A^i$ for*[6] *each $i \in I$. Let $\mathcal{A}^i = (M_c^i \times {A^i}^{H_c^i})$. $S_{\Gamma^c}^i = {\mathcal{A}^i}^{L^i}$ denotes the set of pure strategies of player i in Γ^c.*

Γ^c is a Bayesian game with compact action sets $\mathcal{A}^i$. We skip the subindex c if it is not confusing. A Bayesian Nash equilibrium of Γ^c is a strategy profile $s = (s^1, \dots, s^I)$, where $s^i \in \Delta S_{\Gamma^c}^i$:

$$s^i(l^i) \in arg \max_{(\sigma^i,\rho^i)\in\Delta\mathcal{A}^i} \sum_{l^{-i}\in L^{-i}} \lambda(l^{-i}|l^i)\mathbb{E}_{c(\sigma^i,\sigma^{-i}(l^{-i}))} g^i((\rho^i(h^i), \rho^{-i}(l^{-i}, h^{-i})), (l^i, l^{-i})).$$

for all $l^i \in L^i, i \in I$. Denote the set of Bayes Nash equilibrium payoffs of Γ^c with $BE(\Gamma^c)$.

Definition 2 *An* information structure *on the set $E = \Pi_{i\in I} E^i$, where E^i is compact, is a Borel probability measure μ over E. An element $e = (e^1, \dots, e^I) \in E$ is chosen according to μ, then player i is informed of the component e^i.*

[6] This is without loss of generality since h^i can include m^i.

To put it in another way, an information structure μ on E is a signalling function c with $H_c^i = E^i$ for all i and $c(.) = \mu \in \Delta E$.

Given signalling functions c_1, c_2 we can define a new signalling function. Suppose players first communicate through c_1. After having received their messages from c_1 players communicate through c_2. This sequential communication is equivalent with communicating through the signalling function $c \doteq c_1 \otimes c_2$ which receives messages $(m_{c_1}^i, m_{c_2}^i(.)) \in M_{c_1}^i \times M_{c_2}^{i^{H_{c_1}^i}} = M_c^i$. c selects h_{c_1} according to $c_1(m_{c_1})$ and h_{c_2} according to $c_2(m_{c_2}(h_{c_1}))$ and player i is told $(h_{c_1}^i, h_{c_2}^i) \in H_c^i = H_{c_1}^i \times H_{c_2}^i$. $M_{c_2}^{i^{H_{c_1}^i}}$ is the set of Borel functions from $H_{c_1}^i$ to $M_{c_2}^i$ for all i. A mixed communication strategy $\sigma_c^i(.) : L^i \to \Delta M_c^i$ is equivalent with a pair $(\sigma_{c_1}^i, \sigma_{c_2}^i(.))(.) : L^i \to \Delta M_{c_1}^i \times (\Delta M_{c_2}^i)^{H_{c_1}^i}$ behavioral communication strategy by assuming perfect recall. The construction of new signalling functions can be extended to the product or sequence of infinitely many signalling functions[7] $c = \bigotimes_{t=1}^{\infty} c_t$ without any problem with the specification of the mixed strategies and without the violation of Kuhn's Theorem (1953). Given an infinite sequence of signalling functions we say that the communication is essentially terminated at time T if the communication history is $h_c = (h_{c_1}, \ldots, h_{c_T}, \ldots)$ and ρ^i is constant over the cylinder generated by $(h_{c_1}^i, \ldots, h_{c_T}^i)$ for all $i \in I$.

Let $\mu \in \Delta(\Pi_{i \in I} S_{\Gamma^c}^i)$ an information structure and denote by $(\Gamma^c)^\mu \doteq \Gamma^{\mu \otimes c}$ the following Bayesian game:

1. player i learns his type $l^i \in L^i$ for all $i \in I$ as in Γ
2. $(m_c, \rho_c)(.) = ((m_c^1, \rho_c^1)(.), \ldots, (m_c^I, \rho_c^I)(.)) \in S_{\Gamma^c}$ is selected according to μ
3. player i learns $(m_c^i, \rho_c^i)(.) \in S_{\Gamma^c}^i$
4. players send private messages $m_c' = (m_c'^1, \ldots, m_c'^I)$ to c
5. players receive private messages h_c^i from c, where $h_c = (h_c^1, \ldots, h_c^I)$ is chosen according to $c(m_c')$
6. players choose actions in Γ

Players receive pure strategy recommendations $s^i = (m^i, \rho^i)(.) \in S_{\Gamma^c}^i$ from μ after they have learnt their types, that is a pure strategy in the Bayesian game $(\Gamma^c)^\mu$ of player i is $f^i : L^i \times S_{\Gamma^c}^i \to \mathcal{A}^i$. For every $\mu \in \Delta S_{\Gamma^c}$ and $f \in S_{\Gamma^{\mu \otimes c}}$ there is an induced distribution $P_f^\mu \in \Delta(L \times S_{\Gamma^c} \times H_c \times A)$. Let $f^i = id^i$ if players obediently follow μ that is, for all l^i and all $s^i \in \operatorname{supp} \mu$, $id^i(l^i, s^i) = s^i(l^i)$.

[7]The extension of Kuhn's Theorem for countable message spaces was proposed by Wolfe (1955). More generally see it in Aumann (1964).

Definition 3 *An information structure μ is a correlated equilibrium distribution of Γ^c iff id is a Bayesian Nash equilibrium of $\Gamma^{\mu\otimes c}$.*

Denote $CE(\Gamma^c) \subset \Pi_{i\in I}\mathbb{R}^{|L^i|}$ the set of correlated equilibrium payoffs of Γ^c. Then

$$\cup_{\mu\in\Delta S_{\Gamma^c}} BE(\Gamma^{\mu\otimes c}) = CE(\Gamma^c).$$

Alternatively, one can think of μ as a correlated c-protocol. Let $\sigma_{s^i}^{-i}(.)$ denote player i's belief about the communication strategies of players $-i$ given s^i, that is the marginal of $\mu(.|s^i)$ on $\Pi_{j\in -i}{M^j}^{L^j}$. Also denote $\rho_{s^i}^{-i}(.)$ the belief of player i about the decision rules of $-i$ given s^i, that is the marginal of $\mu(.|s^i)$ on $\Pi_{j\in -i}(H^j \times A^j)^{L^j}$ for some $s^i \in \operatorname{supp}\mu$. μ is a correlated equilibrium distribution of Γ^c if for all strategy profile $s = (s^1, \ldots, s^I) \in \operatorname{supp}\mu$ and for all $l^i \in L^i, i \in I$,

$$s^i(l^i) \in arg \max_{(\sigma^i,\rho^i)\in\Delta\mathcal{A}^i} \sum_{l^{-i}\in L^{-i}} \lambda(l^{-i}|l^i)\mathbb{E}_{c(\sigma^i,\sigma_{s^i}^{-i}(l^{-i}))} g^i(\rho^i(h^i), \rho_{s^i}^{-i}(l^{-i}, h^{-i}), (l^i, l^{-i})).$$

The set $Q = \{q|q : L \to \Delta A\}$ of signalling functions are called the set of canonical communication devices for games with types space L and action sets A. For each $q \in Q$ and λ there is a corresponding distribution $P_{\lambda,q} =\in \Delta(L \times A)$, where $P_{\lambda,q}(l, a) = \lambda(l)q(l)(a)$. We write simply P_q when λ is clear from the context. Denote

$$g^i[q|l^i] = \sum_{l^{-i}} \lambda(l^{-i}|l^i) \sum_a q(l^i, l^{-i})(a)g^i(a, (l^i, l^{-i})) = \sum_{l-i}\sum_a P_q(l^{-i}, a|l^i)g^i(a, (l^i, l^{-i})).$$

the expected payoff of player i of type l^i when all the players are sincere and obedient and the mediator is randomizing according to q.

Definition 4 *q is a canonical communication equilibrium of Γ if and only if all the players being sincere and obedient forms a Bayesian Nash equilibrium of Γ^q:*

$$g^i[q|l^i] \geq \sum_{l^{-i}} \lambda(l^{-i}|l^i) \sum_a q(l'^i, l^{-i})(a)g^i(\rho^i(a^i), a^{-i}, (l^i, l^{-i}))$$

for all i, l^i, l'^i and for all $\rho^i : A^i \to A^i$. Let $ME(\Gamma) \subset \Pi_{i\in I}\mathbb{R}^{|L^i|}$ be the set of communication equilibrium payoffs. In short:

$$\cup_{q\in Q} BE(\Gamma^q) = ME(\Gamma).$$

Let $\mathcal{G}(q) = \{\Gamma|q$ is communication equilibrium of $\Gamma\}$.

Notice that for any signalling function c we have $BE(\Gamma^c) \subseteq ME(\Gamma)$.

Definition 5 *A correlated $c-$protocol μ mimicking q is* secure[8] *or* universal *iff whenever q is a communication equilibrium of Γ then μ is a correlated equilibrium distribution of Γ^c.*

[8]See Gossner (1998) for the characterization of uncorrelated secure protocols for games with complete information.

Now consider the interim public two stage cheap talk extension of a finite Γ, $ext\Gamma \doteq \Gamma^c$ with $c = c_{public} \otimes c_{public}$, where $c_{public} : \mathbb{N}^I \to (\mathbb{N}^I)^I$ such that $c(n_1, \ldots, n_I) = ((n_1, \ldots, n_I), \ldots, (n_1, \ldots, n_I))$. That is after the message profile $(n_1, \ldots, n_I) \in \mathbb{N}^I$ each player receives $(n_1, \ldots, n_I)$, thus the messages are public:

1. players learn their types $l \in L$ as in Γ
2. players send messages to each other simultaneously publicly for two stages from message sets $M^i_{c_{public}}$
3. players choose actions in Γ

Forges established the following equivalence depending on the number of players and the message spaces $M^i_{c_{public}}$ available:

Proposition 1 *Forges (1990): For $|I| > 3$ and $M^i_{c_{public}} = \mathbb{N}$ for any q there is a $\mu(q) \in \Delta S_{ext\Gamma}$ secure correlated protocol mimicking q. The same is true for $|I| = 3$ if $M^i_{c_{public}} = [0, 1]$. As a consequence:*

$$ME(\Gamma) = CE(ext\Gamma)$$

Notice that $\mu(q)$ can be chosen independently of the players preferences g^i in Γ and also independently of λ. $\mu(q)$ only depends on the support of q and on the distributions $q(l)$.

The following definition gives us the payoff vectors that are interim individually rational for the players. The level of these payoffs and the corresponding strategies can serve as punishments in case of deviations.

Definition 6 *Forges (2006): A payoff vector $g^i[q|l^i]_{l^i \in L^i}$ payoff vector is (strictly) interim individually rational for player $i \in I$ (or interim supportable with (strict) punishment) if there is a system of distributions $q^{-i} : L^{-i} \to \Delta A^{-i}$ such that for all $l^i \in L^i$ and $a^i \in A^i$,*

$$g^i[q|l^i] \geq (>) \sum_{l^{-i}} \lambda(l^{-i}|l^i) \sum_{a^{-i}} q^{-i}(l^{-i})(a^{-i}) g^i(a^i, a^{-i}, (l^i, l^{-i})).$$

Let $(S)INTIR(\Gamma)$ be the set of payoffs $g[q]$ in $\Pi_{i \in I} \mathbb{R}^{|L^i|}$, that are (strictly) interim individually rational for every player. Let $\underline{g}^i(l^i)$ the interim individually rational level for player i of type l^i.

In words, suppose at the beginning of a procedure player i of type l^i expects that $-i$ choose actions according to $\sum_{l^{-i}} \lambda(l^{-i}|l^i) q^{-i}(l^{-i})$. Then i is not willing to participate in this procedure if it offers a lower expected payoff than his interim individually rational level at type l^i. This is because there exists an action b^i

with which he can guarantee his interim individually rational level. Notice that i has to know q^{-i} to choose his action b^i which offers him at least his interim individually rational level. Thus the order of the quantifiers in the definition is important. In our construction, player i will not deviate from the equilibrium path because he expects that in case of deviation $-i$ follow q^{-i} to choose actions. Notice also, that the definition requires that i has no more information[9] than his type l^i. It has to be clear then that $ME(\Gamma) \subseteq INTIR(\Gamma)$, see Lemma 3 later for a constructive proof.

Let the universal interim infinite cheap talk extension of a finite Γ,

$$\bigotimes_{t=1}^{\infty} c_{public} = c_{public}^{\infty}$$

Let $EXT\Gamma = \Gamma^{c_{public}^{\infty}}$ be the following extended game:

1. players learn i learns his type $l^i \in L^i$ for all $i \in I$ as in Γ
2. players send messages to each other simultaneously at every time t (possibly infinitely long) from message sets $\mathbb{N}$
3. players choose actions in Γ

Theorem 1 *For any finite Γ with $|I| \leq 3$ and $t \in \mathbb{N}$:*

$$ME(\Gamma) \cap SINTIR(\Gamma) = CE(EXT\Gamma) \cap SINTIR(\Gamma)$$

if $t \in [0, 1]$ or $|I| > 3$:

$$ME(\Gamma) = CE(EXT\Gamma)$$

The communication essentially terminates in finite time with probability 1.

In words, for any finite Γ, if players communicate in discrete time and can use only countable message spaces, then any communication equilibrium q for which $g[q] \in SINTIR(\Gamma)$ there is a $\mu(q, \Gamma)$ over the set of strategy profiles of $EXT\Gamma$ such that:

1. $BE((EXT\Gamma)^{\mu(q,\Gamma)}) \supseteq BE(\Gamma^q)$. Notice that in $EXT\Gamma$ players can communicate directly. As a consequence the extended game can have equilibrium payoffs which are not equilibrium payoffs of Γ^q

[9] The argument for procedures promising strictly interim individually rational payoffs could be extended to games with information structures. It would require that player i's expected payoff after receiving his information l^i, e^i should be strictly interim individually rational for any e^i.

2. id is Bayesian Nash equilibrium of $(EXT\Gamma)^{\mu(q,\Gamma)}$ and $P_q(l,a) = P_{id}^{\mu(q,\Gamma)}(l,a)$ for all $(l,a) \in L \times A$. Thus achieves the same payoffs as q.

3. although μ depends on (q,Γ), the extension EXT is independent of (q,Γ), that is universal in the sense of Forges (1990).

Moreover, if players can communicate in continuous time, the same statements are true for *any* q communication equilibrium and the generating protocols are secure.

We give the main idea of the proof in the next section. The detailed, constructive proof can be found in the Appendix.

3 Proof

3.1 Mimicking with Short Universal Mechanism

A basic property of a (possibly correlated) protocol which wishes to achieve the same effect as some q is as follows. Since players' messages are public we simply write $H^i_{c_{public}} = H$

Definition 7 $\mu \in \Delta S_{\Gamma^c}$ *mimics* $q \in Q$ *iff for all* i, l^i, *all* $s^i = (m^i, \rho^i)(.) \in \operatorname{supp} \mu$ *and all* $(l^i, s^i, h, .) \in \operatorname{supp} P^\mu_{id}$:

1. $P^\mu_{id}(l^{-i}, a | l^i, s^i) = P_q(l^{-i}, a | l^i)$,

2. $P^\mu_{id}(l^{-i}, a^{-i} | l^i, s^i, h) = P_q(l^{-i}, a^{-i} | \rho^i(l^i, h), l^i)$.

Lemma 1 *For any* $q \in Q$ *and for any finite* Γ *there is a* $\mu(q) \in \Delta S_{ext\Gamma}$ *which mimics* q. *Moreover* $M^i_{c_{public}}$ *can be chosen to be countable.*

It is easy to see that, if q is a *communication equilibrium* of Γ and players $-i$ play according to id^{-i} then after the recommendation $s^i = (m^i, \rho^i)(.)$

1. player i of type l^i is always better off communicating according to $m^i(l^i)$ than according to $m^i(l'^i)$

2. $m^i(.)$ however, is not necessarily the optimal communication strategy.

3. given that i of type l^i has communicated according to $m^i(l^i)$ and the history h so far, it is optimal to choose his action according to $\rho^i(l^i, h)$, that is the action recommended, corresponding to the realized history of communication.

It must be clear that ones $\mu(q)$ mimics q it is enough to concentrate on deterring players from *profitable deviations during the communication* to maintain equilibrium. If i of type l^i pretends type l'^i, the condition on communication equilibrium ensures that i cannot be better off with such a deviation in any procedure which mimics q.

Definition 8 *A deviation f^i from id^i is a* strange deviation *of type l^i if there is an $s^i = (m^i, \rho^i)(.) \in \operatorname{supp} \mu$ for which $f^i(l^i, s^i) = (m'^i, \rho'^i)$ and there is no l'^i such that $m^i(l'^i) = m'^i$. A deviation is strange if there is a type for which it is strange.*

3.2 Punishment

For $j \in I$ denote an $|I|$ element partition of $S^j_{\Gamma^c} \times H$ by $P^j = (P^j_1, \ldots, P^j_I)$. We give the interpretation that for a recommendation and history of communication pair j receives in P^j_i, j thinks that i was deviating. Let $D_i = \{(s, h) \in S_{\Gamma^c} \times H | \forall j \in -i \; (s^j, h) \in P^j_i\}$. That is, for recommendations and histories in D_i all the players in $-i$ think that i was deviating. Of course, we want that if nobody deviates, the probability of D_i is 0 for all $i \in I$. Let $D^*_i(l^i, f^i) = \{(s, h) \in D_i | \exists l^{-i} : P^\mu_{(f^i, id^{-i})}(l^{-i}|l^i, s^i, h) \neq \lambda(l^{-i}|l^i)\}$ be the set of histories, where after detected strange deviation f^i, player i of type l^i can update his prior.

Definition 9 *Given μ and a system of partitions $(P^j)_{j\in I}$ such that $P^\mu_{id}(D_i) = 0$ for all i, a strange deviation $f^i \neq id^i$ of type l^i is* punishable *with probability at least $1 - z$ if:*

1. *it is* detectable *with probability at least $1 - z$: $P^\mu_{(f^i, id^{-i})}(D_i | l^i, l^{-i}, s^i) > 1 - z$ for all $l^{-i} \in \operatorname{supp} \lambda(.|l^i)$,*
2. *i* can update his prior *in D^i with probability at most z: $P^\mu_{(f^i, id^{-i})}(D^*_i(l^i, f^i) | l^i, s^i) < z$.*

*We say, that μ is $z-$*weakly secure *if there is $(P^j)_{j\in I}$ such that any strange deviation of any player of any type is punishable with probability at least $1 - z$.*

Lemma 2 *For any q and $z > 0$ there is $\mu(q, z) \in \Delta S_{ext\Gamma}$ mimicking q and strange deviations are detectable with probability at least $1 - z$.*

Proof: See the proof in the Appendix. ■

The connection between security and weak security can be partially understood with the following 2 Lemmas.

Lemma 3 *$M(\Gamma) \subseteq INTIR(\Gamma)$.*

Proof: Given a communication equilibrium q and a type l^i then for any $b^i \in A^i$ and $l'^i \in L^i$

$$g^i[q|l^i] = \sum_{l^{-i}} \lambda(l^{-i}|l^i) \sum_a q(l^i, l^{-i})(a) g^i(a, l^i, l^{-i}) \geq$$

$$\geq \sum_{l^{-i}} \lambda(l^{-i}|l^i) \sum_{a^i, a^{-i}} q(l'^i, l^{-i})(a^i, a^{-i}) g^i(b^i, a^{-i}, l^i, l^{-i}) =$$

$$\sum_{l^{-i}} \lambda(l^{-i}|l^i) \sum_{a^{-i}} q(l'^i, l^{-i})(a^{-i}) g^i(b^i, a^{-i}, l^i, l^{-i})$$

So let $q^{-i}(l^{-i})(a^{-i}) = q(l'^i, l^{-i})(a^{-i})$ for some $l'^i \in L^i$ as punishment. ■

Suppose μ mimics q and player i of type l^i deviates strangely during the communication. If this misbehavior is detected by players $-i$ before i would have learnt any new information about l^{-i} that is the recommendation, communication history pair is in $D_i \setminus D_i^*(l^i, f^i)$ then players $-i$ can choose an action according to $q(l'^i, l^{-i})(a^{-i}) = q^{-i}(l^{-i})(a^{-i})$ for some l'^i independently of player i's realized type l^i. It does not matter what constant action b^i is chosen by i, she is not better off than $g^i[q|l^i]$ if q is a communication equilibrium. If $g[q] \in SINTIR(\Gamma)$ then i is strictly worse off. In short, players $-i$ can punish deviations of i if the deviation occurs and detected until i of type l^i believes $\lambda(.|l^i)$ as the second property of weak security requires. Note that player i cannot be punished with the profile given above if i has already got to know his action a^i that is in $D_i^*(l^i, f^i)$. This is simply because i can condition his action on a^i with some function $r : A^i \to A^i$ which is in general strictly better than some constant action b^i. Moreover, i knows, that a^i came from q, thus he can updated his information about l^{-i}.

Definition 10 *$\mu \sim \mu'$ iff $P_{id}^{\mu} = P_{id}^{\mu'}$*

Definition 11 *$g[q]$ is $z-$interim supportable iff for all i, l^i*

$$zW^i + (1-z)[zW^i + (1-z)\underline{g}^i(l^i)] \leq g^i[q|l^i],$$

where $W^i = \max_{l \in L, \rho \in \Delta A} g^i(\rho, l)$.

Let $\mathcal{G}(q,z) = \{\Gamma \in \mathcal{G}(q) | g[q] \ z-interim\ supportable\}$

The rationale of this definition comes from the following Lemma:

Lemma 4 *If μ is z-weakly secure c-protocol which mimics q then for any $\Gamma \in \mathcal{G}(q,z)$ there is a $\mu' \sim \mu$ correlated equilibrium distribution of Γ^c if $|I| = 2$. If $|I| = 3$ there exists μ' correlated equilibrium distribution of $\Gamma^{c \otimes c_{public}}$ where the last stage of communication is used only in case of deviations under c, that is we can write that $\mu' \sim \mu$.*

Proof: The construction of such a μ' is simple in the case of 2 players. One has to define ρ^j on P_i^j to be the appropriate punishments in Γ. Then a deviating player gets at most W^i with probability at most z, when the deviation was not detected. With probability at most $(1-z)z$ he gets at most W^i again, if the deviation was detected but the deviator updated his prior. Finally receives his punishment payoff with probability at least $(1-z)(1-z)$.

The 3 players' case is a bit more delicate, since players $-i$ has to correlate their punishment actions. It is simple if we allow the players to communicate in pairwise channels, however it can be achieved without this assumption. See the solution in the Appendix. ■

If the number of players is 2 and it is known or it is predictable with high probability for player i when the communication terminates, there are games[10] Γ, where certain types of i have incentives to deviate in the very last stage, where updating happens. Doing so, i learns his action while $-i$ does not get the proper information to be able to calculate his action. Even if the deviation is detected, $-i$ cannot punish i.

Our solution for this problem is constructing a protocol where players cannot predict when the communication terminates. More precisely, players do not know in advance the stage at which they receive the relevant information from which the actions are calculable, that is when updating their believes would be possible. This can be achieved with a possibly infinitely long communication and random termination times with uniformly small probabilities in each stage. This uncertainty, of course, is introduced by the correlation device. The main idea of our construction is that we hide the mimicking protocol among uninformative ones. Doing so we can get a $z-$weakly secure protocol for any $z > 0$.

3.3 A Long Universal Mechanism with Punishment

Let $\mu(u,z) \in \Delta S_{ext\Gamma}$ be the mediator which mimics $u : L \to \Delta A$ such that for any $l, l' \in L$ and $a, a' \in A$ $u(l)(a) = u(l')(a')$ and strange deviations are detectable with probabiliy at least $1-z$. That is u selects uniformly and independently actions for any type-profile.

Lemma 5 *$\mu(q,z), \mu(u,z) \in \Delta S_{ext\Gamma}$ mimicking q, u can be chosen in a way that for all $i, l^i, s^i \in S_{ext\Gamma}$ for all $h = (h_1, h_2) \in H^2_{c_{public}} = \mathcal{Q} \cup \mathcal{U}$, where $\mathcal{Q} \cap \mathcal{U} = \emptyset$*

1. $\mu(q,z)(s^i) = \mu(u,z)(s^i)$
2. $P_{id}^{\mu(q,z)}(h_1|l^i, s^i) = P_{id}^{\mu(u,z)}(h_1|l^i, s^i)$

[10]This problem does not arise if only one of the players has actions to take, see the construction in Forges (1985).

3. $P_{id}^{\mu(q,z)}(\mathcal{Q}|l^i, s^i) = 1, P_{id}^{\mu(u,z)}(\mathcal{U}|l^i, s^i) = 1$

Proof: See the proof in the Appendix ■

Suppose $s \in S_{ext\Gamma}$ is chosen according to $\mu(q,z)$ with probability z and according to $\mu(u,z)$ with probability $1-z$. It follows from point 1 and 2 in the Lemma that player i receiving recommendation $s^i \in S_{ext\Gamma}$ still believes with probability $z, 1-z$ that s^i was chosen according to $\mu(q,z)$ or $\mu(u,z)$. The same is true after observing h_1 in the first stage of communication. However, after the second stage, according to point 3, players know for sure, whether s was chosen according to $\mu(q,z)$ or $\mu(u,z)$. The main idea in the construction of $z-$weakly secure $\mu(z) \in \Delta S_{EXT\Gamma}$ which mimics q is as follows:

1. $\mu(z)$ chooses a time t^* randomly according to a geometric distribution with parameter z

2. for all $t \neq t^*$ $\mu(z)$ sends recommendations $s_t^i \in S_{ext\Gamma}$ to the players according to $\mu(u,z)$ which does not convey any information, while for t^* it randomizes according to $\mu(q,z) \in \Delta S_{ext\Gamma}$

In equilibrium:

1. players communicate according to s_t^i in stages $2t-1, 2t$

2. after each even stage players can decide if the two stages corresponded to t^* or not, but never before exchanging their messages in that stage (as consequence of Lemma 3 players can check if the two stage history is in $\mathcal{Q}$ or in $\mathcal{U}$).

3. after stage $2t^*$ or in case of detected deviation players decide to play in Γ and play according to the recommendation of $\mu(q,z)$ or choose their punishment actions

Lemma 6 *$\mu(z)$ is z-weakly secure and mimics q.*

Proof: It is obvious that μ mimics q. Property 1 of weak security is satisfied, that is strange deviations are detectable with probability at least $1-z$, since $\mu(q,z), \mu(u,z)$ satisfies this requirement. For property 2, it is clear that if a deviation is detected at a stage, where $\mu(z)$ randomized according to $\mu(u,z)$ the deviator cannot update his prior. The probability that a deviation f^i finds t^* and it is not detected in stages $2t-1, 2t < 2t^*$ is clearly less than z. ■

Remark 1 *Notice, that if t^* can be chosen uniformly from the interval $[0,1]$, then any deviation on a countable subset of $[0,1]$ finds t^* with probability 0. For each such deviation at each stage there is a positive probability of being detected and punished. On an infinite countable set the probability of detection is 1. Thus,* *even players who cannot be strictly punished can find profitable deviations*[11] *with probability 0.*

We can prove Theorem 1 as a Corollary of the Lemmas.

Proof of the Theorem: For any Γ and q for which $g[q] \in ME(\Gamma) \cap SINTIR(\Gamma)$ there is a z such that $\Gamma \in \mathcal{G}(q, z)$. By Lemma 2 we have $\mu(q, z)$ correlated protocol mimicking q, where deviations are detectable with probability at least $1 - z$. By Lemma 6 we can construct a $z-$weakly secure protocol $\mu(z)$. Finally by Lemma 4 there is $\mu'(z) \in \Delta S_{EXT\Gamma}$ such that $g[q] \in BE((EXT\Gamma)^{\mu'(z)})$ that is $g[q] \in CE(EXT\Gamma)$. According to the remark, if $t^* \sim U[0, 1]$, the argument above holds for any q for which $\Gamma \in \mathcal{G}(q, 0)$ that is $g[q]$ can be in $INTIR(\Gamma)$. ■

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[11]For $t \in [0, 1]$ one can modify the information structure $\mu(z)$ such that it selects $t^* \in [0, 1]$ according to the uniform distribution. Such an information structure exists by Kolmogorov's existence Theorem (see the construction of such a measure in Billingsley (1995) Theorem 36.2). Notice also, that players' expectations are well defined.

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4 Appendix

We have to prove Lemma 1,2 and 5. We also have to prove Lemma 4 for $|I| = 3$. Clearly, Lemma 1 follows from Lemma 2.

Proof of Lemma 2: $\mu(q, z)$ selects:

1. randomly, uniformly a permutation of the elements of L^i say η^i for each i independently. Let $\eta = (\eta^i)_{i \in I}$.
2. $a_{\eta(l)}$ according to $q(l)$
3. randomly, uniformly 2 permutations of the elements of A^i for each $l \in L$ and for each i independently say $(\phi^i_l, \psi^i_l)_{l \in L},$. Let $\phi_l = (\phi^i_l)_{i \in I}, \psi_l = (\psi^i_l)_{i \in I}$. Let
$$b_{\eta(l)} = \psi_{\eta(l)}(\phi_{\eta(l)}(a_{\eta(l)}))$$
4. codes randomly, uniformly from $Z \subset \mathbb{N}$, where $\frac{1}{|Z|} < z$: $k^i(\eta(l), b^i)$ for all i, l and all $b^i \in A^i$ and $K^i(\eta(l), \psi^i)$ for all i, l and all ψ^i with the restriction that for all $i, j \in I$:
$$k^i(\eta(l), b^i_{\eta(l)}) = k^j(\eta(l), b^j_{\eta(l)})$$
5. if $|I| = 2$,$\mu(q, z)$ sends for all $l \in L$, $s^i =$
$$\eta^i, \phi^i_{\eta(l)},$$
$$k^i(\eta(l), .), K^i(\eta(l), .),$$
$$b^{-i}_{\eta(l)}, k^{-i}(\eta(l), b^{-i}_{\eta(l)}),$$
$$\psi^{-i}_{\eta(l)}, K^{-i}(\eta(l), \psi^{-i}_{\eta(l)}),$$
to player i.

After receiving s^i, the communication unfolds as follows if $|I| = 2$. Let for all i, player i of type l^i announce publicly $m^i_1(l^i) = \eta^i(l^i)$ in stage 1.

The communication protocol in stage 2 prescribes that after the announcement $h_1 = \eta(l)$ in stages 1 player i publicly announces $m^i_2(h_1)(l^i) =$:
$$b^{-i}_{\eta(l)}, k^{-i}(\eta(l), b^{-i}_{\eta(l)}),$$
$$\psi^{-i}_{\eta(l)}, K^{-i}(\eta(l), \psi^{-i}_{\eta(l)}).$$
player i's decision rule is defined as:
$$\rho^i(s^i, h) = \phi^{i^{-1}}_{\eta(l)}(\psi^{i^{-1}}_{\eta(l)}(b^i_{\eta(l)})) = a^i_{\eta(l)}.$$
which was selected according to $q(l)$. Formally, we can define P^j_i naturally $\{(s^j, h) | k^j(\eta(l), b^j) \neq k^j \ or \ K^j(\eta(l), \psi^j) \neq K^j\}$, where $m^i_2 = (b^j, \psi^j, k^j, K^j)$ is the announcement of player i in stage 2 and $h_1 = \eta(l)$. In words, player j thinks that i deviated, if the codes k^j, K^j announced by i do not coincide with the value of $k^j(.,.), K^j(.,.)$ at $\eta(l), b^j, \psi^j$.

If $|I| = 3$ we impose the following modifications on $\mu(q, z)$ and on the communication.

5. if $|I| = 3$,$\mu(q,z)$ sends for all $l \in L$, $s^i =$

$$\eta^i, \phi^i_{\eta(l)},$$

$$k^i(\eta(l), .), K^i(\eta(l), .),$$

$$b^{-i}_{\eta(l)}, k^{i \ mod \ 3 \ +1}(\eta(l), b^{i \ mod \ 3 \ +1}_{\eta(l)}),$$

$$\psi^{-i}_{\eta(l)}, K^{i \ mod \ 3 \ +1}(\eta(l), \psi^{i \ mod \ 3 \ +1}_{\eta(l)}),$$

to player i.

Let for all i, player i of type l^i announce publicly $\eta^i(l^i)$ in stage 1 as in the case of 2 players. Then if $|I| = 3$, players are prescribed to publicly announce in stage 2 $m_2^i(h_1)(l^i) =:$

$$b^{i \ mod \ 3 \ +1}_{\eta(l)}, k^{i \ mod \ 3 \ +1}(\eta(l), b^{i \ mod \ 3 \ +1}_{\eta(l)}).$$

$$\psi^{i \ mod \ 3 \ +1}_{\eta(l)}, K^{i \ mod \ 3 \ +1}(\eta(l), \psi^{i \ mod \ 3 \ +1}_{\eta(l)}),$$

player i's decision rule is:

$$\rho^i(s^i, h)(l^i) = \phi^{i-1}_{\eta(l)}(\psi^{i-1}_{\eta(l)}(b^i_{\eta(l)})) = a^i_{\eta(l)}.$$

which was selected according to $q(l)$. That is, 1 sends to 2, 2 to 3 and 3 to 1 the relevant information. Notice also that, players only know one code and their own code function. P^j_i can be defined similarly to the case of 2 players with the following modification. Suppose that 1 deviates and 2 detects it because 1 have sent a wrong code. 3 also knows that 1 was deviating because 3 knows $b^2_{\eta(l)}$. ■

Proof of Lemma 4: Fix a $\Gamma \in \mathcal{G}(q,z)$, and $(q^{-i}(l^{-i})_{l^{-i} \in L^{-i}})_{i \in I}$ punishment distributions. Let μ' select additionally to μ for all $i \in I$ and $l^{-i} \in L^{-i}$

1. (η^i) permutations of L^i

2. $p^{-i}_{\eta^{-i}(l^{-i})} \in A^{-i}$ according to $q^{-i}(l^{-i})$

3. randomly and uniformly a permutation $\varphi^j_{l^{-i}}$ of the elements of A^j for $j \in -i$ and let

$$d^j_{\eta^{-i}(l^{-i})} = \varphi^{j \ -1}_{\eta^{-i}(l^{-i})}(p^j_{\eta^{-i}(l^{-i})})$$

Then μ' sends $\eta^{j_1}, d^{j_2}_{\eta^{-i}(l^{-i})}, \varphi^{j_1}_{\eta^{-i}(l^{-i})}$ to player $j_1 \neq j_2 \in -i$.

In case $j_1, j_2 \in -i$ detected that i was deviating in the communication during c, then j_1, j_2 of type l^{j_1}, l^{j_2} publicly announces $\eta^{j_1}(l^{j_1}), (d^{j_2}_{\eta^{-i}(l^{-i})})_{l^{-i} \in L^{-i}}, \eta^{j_2}(l^{j_2}), (d^{j_1}_{\eta^{-i}(l^{-i})})_{l^{-i} \in L^{-i}}$ and calculate and play the punishment action $p^{-i}_{\eta^{j_1}(l^{j_1}), \eta^{j_2}(l^{j_2})}$ which was selecting according to $q^{-i}(l^{-i})$. Notice that after the public communication of j_1, j_2 in the last stage, player i of type l^i believes $\lambda(l^{-i}|l^i)$ about l^{-i} and $\sum_{l^{-i} \in L^{-i}} \lambda(l^{-i}|l^i) q^{-i}(l^{-i})(p^{-i})$

about p^{-i}. ■

Proof of Lemma 5: The construction of $\mu(u,z)$ is the same as that of $\mu(q,z)$. We have to modify step 2 and step 4. In step 2, $a_{\eta(l)}$ is chosen according to $u(l)$, that is uniformly from the set of action profiles. In step 4, the code functions k^i are chosen to satisfy the following conditions:

1. $$k^i(\eta(l), b^i_{\eta(l)}) \neq k^j(t, \eta(l), b^j_{\eta(l)})$$

2. if $|I| = 2$ we require that there are $b^1 \in A^1, b^2 \in A^2$ such that:
$$k^1(\eta(l), b^1_{\eta(l)}) = k^2(\eta(l), b^2),$$
$$k^2(\eta(l), b^2_{\eta(l)}) = k^1(\eta(l), b^1),$$

3. if $|I| = 3$ there exists $b^1 \in A^1, b^2 \in A^2, b^3 \in A^3$:
$$k^3(\eta(l), b^3_{\eta(l)}) = k^2(\eta(l), b^2),$$
$$k^1(\eta(l), b^1_{\eta(l)}) = k^3(\eta(l), b^3),$$
$$k^2(\eta(l), b^2_{\eta(l)}) = k^1(\eta(l), b^1).$$

We can thus define $\mathcal{Q} = \{(h_1, h_2) \in H^2 | k^i = k^j, \forall i, j \in I\}$ and $\mathcal{U} = \{(h_1, h_2) \in H^2 | \exists i, j \in I : k^i \neq k^j, \}$, where $(k^i)_{i \in I}$ are the codes corresponding to $(b^i)_{i \in I}$ announced in stage 2. Clearly $\mu(u,z), \mu(q,z), \mathcal{Q}, \mathcal{U}$ satisfy conditions 1,2 and 3 in the Lemma. ■