Signal Extraction and Hyperinflations with a Responsive Monetary Policy*

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ABSTRACT

This paper develops a multi-period extension of the Lucas (1972) overlapping generations "island" model with endogenous monetary policy (based on the minimization of a loss function over inflation and output deviations) and stochastic realization of the "allocation" of the young people across the two islands. These allocation realizations are interpreted as output shocks (since only the young people produce). The paper examines two cases: the certainty case when the exact monetary policy is known to the young, and uncertainty case where the young receive only a mixed signal of the output shock and the monetary policy weights through the price (the signal extraction problem). In the certainty case, the neutrality result holds. In the uncertainty case, even monetary shocks have real effects as a result of the signal extraction problem. After characterizing the resulting price function by its constant elasticity to the signal, we derive values of this elasticity and the monetary policy weights such that hyperinflations will develop. We find that for certain weights, hyperinflations can develop even when the price function is concave in the signal. Finally, we formulate a particular convex case of the price function (making distributional assumptions) to analyze the price and monetary policy examples and statics as functions of the weights on the inflation and output deviation terms.

Keywords: Rational expectations, Neutrality of Money, Signal Extraction Problem, Loss function, Hyperinflations, High inflations

Introduction

Models of high inflations and hyperinflations can be separated into two groups. The first group examines the hyperinflationary episodes of the European economies in the first half of the twentieth century, which generally occurred in the framework of economic crises following major wars. The most infamous episode of such hyperinflations occurred in Germany in the 1920s, when monthly inflation reached 3.25×10^6. Hungary also

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experienced hyperinflation between the two world wars, when inflation rates were as high as 4.19×10^{16} per month. Models of these episodes are generally explanatory in nature. The most important models are Cagan's seminal 1956 work, building a model of adaptive expectations. Sargent and Wallace (1973) developed an elegant rational expectations theory of hyperinflations, whereby they show that under certain conditions (such as the feedback from inflation to future rates of money creation), Cagan's adaptive expectations can be rational as well. They make the interesting argument that high rates of inflation cause high rates of money creation, and not the other way around, as was believed by many researchers at the time. Sargent (1981) further contributed to the understanding of such hyperinflations with his detailed empirical study of four central European countries' experience with "big inflations".

The second group of high inflation models examines inflationary outcomes as functions of various types of monetary policy rules in the context of developed economies, estimating rule parameters and making policy recommendations. The most influential papers in this group were written by Taylor (1993,1997) and Woodford (1999). These papers are mostly focused on policy implications as opposed to the explanatory focus of the first group of models. The model of this paper belongs to this second group.

The modeling context of this paper is the "island" model. The "island parable" literature developed in the early 1970s, first invented by Phelps (1968) and further developed by Lucas (1972). The island framework allows us to study situations where the *signal extraction problem* faced by fully rational agents results in *real effects* of monetary policy, while fully consistent with the classical idea of the neutrality of money[1]. Lucas developed an overlapping generations model where the elasticity of the price function with respect to the signal is strictly positive, but less than unity. Lucas assumes the rate of money creation is strictly exogenous. Further, Lucas's model extends to two periods only and is not aimed at studying the evolution of the price level over time.

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[1] The quantity theory of money incorporates the idea of the "neutrality of money". This theory claims that changes in the money supply result in proportional changes in the price level. Further, changes in the *level* of the money supply cannot affect the *real* side of the economy (such as GDP and employment). If prices grow at the same rate as the money supply, the real balances remain constant over time.

This paper ties the high-inflation and the "island" literature together by developing a multi-period "island" model where an active monetary authority determines the rate of money creation based on the minimization of a loss function (a weighted sum of squared inflation and output deviations). The allocation of the young people across the two islands is exogenous, and follows a stochastic dynamic AR(1) process. Without loss of generality, the models examines the first island's economy only. Since only the young people produce, the realizations of the allocation variable are interpreted as output shocks. As a result of the signal extraction problem, the young people only observe a signal from the price function, which signal is a mixture of the monetary expansion rate and the real shock to the economy (both unknown to the young people). The optimal monetary policy depends on the signal elasticity of the price and the weights on the inflation and output deviations in the loss function. We find the relationship between the loss function weights and the elasticity of the price function such that hyperinflations develop. We find that hyperinflations can develop for concave-in-the-signal price functions as well, for certain values of the loss function weights.

The model in this paper is consistent with rational expectations in the sense of Muth (1961) and the classical quantity theory of money.

The paper is structured as follows. We start by describing the modeling framework, followed by a detailed discussion of the main theoretical model. We develop the monetary policy and derive the inflation rate. We find conditions for hyperinflations to develop. Next, we make specific distributional assumptions to derive a specific price function, which is convex in the signal. We use this example to illustrate the comparative statics of the model, as well as potential outcomes. The last section summarizes the results. Detailed derivations and calculations are shown in the appendices at the end of the paper.

The Model

The modeling framework in this paper is an "island" economy, based on the model of Lucas (1972). This is an overlapping generations model. There are two islands in this economy, with a total of 2K consumers living on the two islands. K consumers are young, and K consumers are old. At time t, fraction θ_t of the young people live on the first island and fraction $1-\theta_t$ live on the second island. Without loss of generality, it is sufficient to analyze equilibrium on the first island. The realization of θ_t is unknown to both generations, and its value must be inferred from the signal through the price function. The old people are allocated across the two islands so as to equate money demand across the islands: K/2 old people live on both islands.

The assignment of activities is as follows. The young people consume and produce at the same time, while the old people only consume. Production takes place in each period, and storage of the consumption good is not possible. Young people produce the single consumption good in the form of home production (according to a linear production function). They consume a fraction of their output, and sell the remaining output to the old for money, in order to secure old-age consumption. Young consumers solve a two-period optimization problem, whereby they decide 1) how much output to produce, 2) how much of the output to consume, and 3) how much of the output to sell to the old for money. The old's decision problem is trivial: they use all their money to buy the good from the young generation, using the money they brought with them from their youth. Since only the young people produce in this model, θ_t is an indicator of *aggregate output* in the economy. Consecutive realizations of θ_t are interpreted as an output shock process.

The government (or monetary authority) plays an active role in this model: money enters the economy as the government distributes money to the old people in each period (in the form of welfare payments), proportional to what they already have, in the spirit of "helicopter money":

$$M_{t+1} = M_t * x_t$$

where x_t is the rate of money creation in period t. The government determines the rate of money creation based on the minimization of a loss function, as shown below.

The two random variables of the model are θ_t, the fraction of young people on a given island at time t, and the weights the government places on the inflation and output deviation terms in the loss function (see below). This can be interpreted as the citizen's uncertainty about the exact monetary policy of their government (Morris and Shin 2005). New values of the random variables are realized each period according to their respective distributions. The weights are chosen by the monetary authority each period. The real shock realizations are unknown to both generations in the given period. All past values of the random variables are public knowledge. Since the weights of the loss function are unknown to them, the citizens also do not know the exact value of x_t. The young people therefore cannot exactly predict the government's choice of the rate of money creation from the minimization of the loss function (the choice problem is described below).

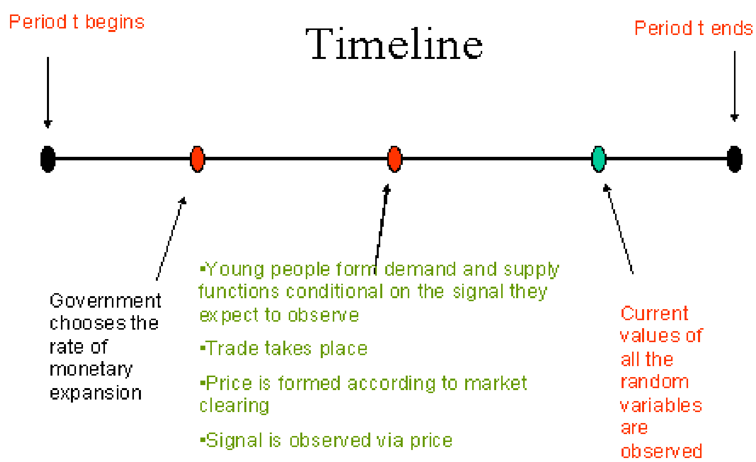

Figure 1: Timing of events in the economy at time t

In the beginning of period t, the government chooses the rate of monetary expansion x_t, based on the minimization of the loss function. Second, the old and the young people form their demand and supply functions based on the maximization of a utility function. For the young people, these demand and supply functions are conditional on the signal they expect to observe through the price. Immediately following this is the period of *trading* between the old and the young people. As a result, the price is formed according to market clearing. The signal $z_t = x_t / \theta_t$ is observed through the price function. Finally, at the end of the period, the contemporaneous values of the random variables are observed, so that they are public knowledge for the generations of the next time period t+1.

It is important to emphasize that the young people can only observe a signal from the price, instead of the exact contemporaneous values of the random variables. Since this signal is a ratio, the young people do not know how much of the price change comes from $x_t = f(\Phi_{t1}, \Phi_{t2})$ via the weight variables in the loss function Φ_{ti}, or the real shock θ_t. This is the *Signal Extraction Problem (SEP).* As a result of SEP, changes in the *monetary* variable x_t (changes in the value of the weight terms) cause *real* changes in the economy. As we will show below, the price function is a function of the signal only, and its evolution depends on the signal elasticity of the price, which we assume to be constant.

The solution of this model is as follows. To begin with, we describe the real shock process and the monetary policy formation. Next, we solve the utility maximization problems of the old and young generations at time t. Then, we examine the certainty (monetary policy is known) and the uncertainty (unknown monetary policy) separately.

The Real Shock Process

We assume that the output shock process (the random allocation variable) follows an AR(1) process.

$$\theta_t = \nu\theta_{t-1} + \epsilon_t \quad \text{where } \epsilon_t \text{ is uniform } (-\eta; \eta)$$

As we will see below, restricting the persistence parameter to equal 1 yields a random walk process.

The Monetary Policy

The goal of the monetary policy is to minimize expected deviations from the inflation and output targets. This policy is essentially the Taylor Rule (Taylor 1993) with a "leaning against the wind" policy. This Taylor rule looks like:

$$r_t = h(x_t) = \phi_{1t}(\pi_t - \bar{\pi}) + \phi_{2t}(N_t - \bar{N})$$

The monetary authority chooses the rate of money expansion x_t to solve:

$$Min_{x_t} \qquad L_t = \phi_{1t}\left[\pi_t(x_t) - \bar{\pi}\right]^2 + \phi_{2t}\left[N_t(x_t) - \bar{N}\right]^2$$

The weights (Φ_{t1}, Φ_{t2}) are chosen by the monetary authority and are unknown to the young people. With the "leaning against the wind" policy, both weights are positive, hence we examine only positive values of the weight terms.

Money Market Equilibrium

There are $K\theta_t$ young people on the first island, and K/2 old people. By money market clearing, the per-young-person money demand is:

$$\lambda_t = \frac{M_{t-1} z_t}{2}$$

where

$$z_t = \frac{x_t}{\theta_t}$$

is the signal that each citizen observes through the price.

The Decision Problems

The decision problem of the old generation is trivial; the old do not produce, and they use all the money they hold to purchase the single consumption good from the young. Since the old's utility is strictly increasing in the consumption good, they spend all their money on the good. The old people's utility maximization problem is:

$$\begin{aligned} Max_{c_t^o} \quad U_t^o\left(c_t^o\right) &= \left(c_t^o\right)^{\alpha} \\ s.t. \quad p_t c_t^o &= M_t \end{aligned}$$

where M_t is the per-old-person money holding at time t.

Hence, the individual old person's demand function becomes:

$$c_t^o = \frac{M_t}{p_t}$$

The old people spend all their money on the consumption good. No inheritance is possible in this model.

Next, we examine the optimization problem of the young generation. The young people decide how much to produce, and how much of this production to consume today or sell to the current old generation for money (in order to secure old-age consumption). The variables of interest are defined in Figure 2.

Variable Definitions

λ_t = per-young-person amount of money at time t

c_t^y = consumption by a young person at period t

n_t^y = production by a young person at period t

p_t = price of the consumption good at period t

c_{t+1}^o = consumption by time t's young generation in period t+1

x_t = Rate of money creation at period t ($M_{t+1} = x_{t+1}M_t$)

Figure 2: Variable definitions

The young consumers' two-period utility maximization problem can be written as:

$$
\begin{aligned}
Max \qquad U(c_t, n_t, \lambda_t) &= (c_t^y)^\alpha - n_t^y + E_t\left[(c_{t+1}^o)^\alpha\right] \\
p_t\left(n_t^y - c_t^y\right) - \lambda_t &\geq 0 \\
c_{t+1}^o &= \frac{\lambda_t x_{t+1}}{p_{t+1}}
\end{aligned}
$$

We assume the following conditions hold:

$$\frac{\partial U}{\partial c_t} > 0 \qquad \frac{\partial^2 U}{\partial c_t^2} < 0 \qquad \frac{\partial U}{\partial n_t} < 0$$

$$0<\alpha<1$$

$$\lim_{c_t \to \infty} \frac{\partial U}{\partial c_t} = 0$$
$$\lim_{c_t \to 0} \frac{\partial U}{\partial c_t} = \infty$$

Assuming general regularity conditions, the first-order conditions become:

$$(1) \qquad \alpha(c_t^y)^{\alpha-1} - \mu_t p_t = 0$$

$$(2) \qquad -1 + \mu_t p_t = 0$$

$$(3) \qquad \alpha E_t\left[\left(\frac{\lambda_t x_{t+1}}{p_{t+1}}\right)^{\alpha-1} \frac{x_{t+1}}{p_{t+1}}\right] - \mu_t = 0$$

$$(4) \qquad p_t(n_t^y - c_t^y) - \lambda_t = 0$$

The demand and supply functions:

$$(1') \qquad c_t^y = (\alpha)^{\frac{1}{1-\alpha}}$$

$$(2') \qquad n_t^y = \frac{\lambda_t}{p_t} + (\alpha)^{\frac{1}{1-\alpha}}$$

$$(3') \qquad E_t\left[\lambda_t^{\alpha-1}\left(\frac{x_{t+1}}{p_{t+1}}\right)^{\alpha}\right] = \frac{1}{\alpha p_t}$$

Using money market clearing results from above, we can rewrite the above as:

$$p_t = \frac{(M_{t-1})^{1-\alpha}}{\alpha(z_t)^{\alpha-1} E_t\left[\left(\frac{x_{t+1}}{p_{t+1}}\right)^{\alpha} \mid z_t\right]}$$

In the following, we describe two cases. The first case shows the outcome if the monetary authority announces the exact value of money creation, i.e. when there is *no signal extraction problem*. The second case analyzes the outcome with the signal extraction problem.

Case 1: When the exact monetary policy is announced

From the Euler equation, we can derive the general form of the price function as:

$$\log p_t = \sum_{k=t}^{\infty} \alpha^{k-t}(1-\alpha)\log\lambda_k - \sum_{s=t+1}^{\infty} \alpha^{s-t}\log x_s - \\ - \sum_{k=t}^{\infty} \alpha^{k-t}\log\alpha$$

Then the neutrality of money results hold:

$$\frac{d\log p_t}{d\log x_t} = 1 \qquad \frac{d\log n_t}{d\log x_t} = 1$$

In order to reach the inflation target, the monetary authority just sets the rate of money creation such that:

$$\bar{x}_{t+1} = 1 + \bar{\pi}$$

Now we turn to the case where the monetary policy is not exactly known.

Case 2: When the exact monetary policy is unknown to the citizens

The following table shows the information structure in the uncertainty case.

Information Structure:			
Who Knows What?			
	Past Values of R.V.'s	Weights in the Loss Function	Current Output Shock
Young People	yes	no	no
Old People	yes	no	no
Monetary Aut.	yes	yes	no

The general form of the price function is (for the detailed derivation of the price function, see Appendix A):

$$p_t = \frac{M_{t-1} z_t}{\alpha^{\frac{1}{1-\alpha}} \Gamma(z_t)} = \left[\frac{M_{t-1}}{\alpha^{\frac{1}{1-\alpha}}}\right] \left[\frac{z_t}{\Gamma(z_t)}\right]$$

How the price function reacts to changes in the value of the signal depends on the signal elasticity of the price, which we assume is constant:

$$\frac{d\log p_t}{d\log z_t} = \varepsilon_{p,z} \text{ constant}$$

Given this price functional form, we can derive the optimal monetary policy as a function of the signal elasticity of price, based on the minimization of the loss function described above.

$$\log x = \frac{\varepsilon\phi_1 \bar{\pi} - \phi_2 (1-\varepsilon) \left[\varepsilon \log\theta - \log \bar{N}\right]}{\phi_1 \varepsilon + \phi_2 (1-\varepsilon)^2}$$

In order to find the elasticity as a function of the loss function weights such that hyperinflations develop, we use Cagan's definition of hyperinflations. Cagan defined hyperinflations as inflation which exceeds 50%.

Plugging the above monetary policy into the expression for the inflation rate:

$$\pi = \frac{\varepsilon\phi_1 \bar{\pi} - \phi_2 (1-\varepsilon)\left[\varepsilon \log\theta - \log \bar{N}\right]}{\phi_1 \varepsilon + \phi_2 (1-\varepsilon)^2} - \varepsilon \log[v]$$

Then we can solve for the condition under which hyperinflations develop for varying values of the inflation and output weights:

$$\varepsilon > \frac{\sqrt{[\phi_1 \bar{\pi} - \phi_2 \log\theta - \phi_2 \log\bar{N} - 0.5\phi_1 + 0.5\phi_2 - \phi_2 \log[v]]^2 - 4[\phi_2 \log\theta - \phi_1 \log[v] + \phi_2 \log[v]][\phi_2 \log\bar{N} - 0.5\phi_2]} \\ -[\phi_1 \bar{\pi} - \phi_2 \log\theta - \phi_2 \log\bar{N} - 0.5\phi_1 + 0.5\phi_2 - \phi_2 \log[v]]}{2[\phi_2 \log\theta - \phi_1 \log[v] + \phi_2 \log[v]]}$$

If we let $\theta = 0.5$ and $v = 1$, and $\pi = \log N = 0.05$ this simplifies to:

- $$\varepsilon > \frac{\sqrt{[(\phi_2 - \phi_1)(0.45) - \phi_2 \log\theta]^2 + 1.8\phi_2^2 \log\theta} - [(\phi_2 - \phi_1)(0.45) - \phi_2 \log\theta]}{2\phi_2 \log\theta}$$

Normalizing, we can show the region of elasticities as a function of the weights such that hyperinflations develop:

Price Elasticities for which hyperinflations develop for $\theta = 0.5$

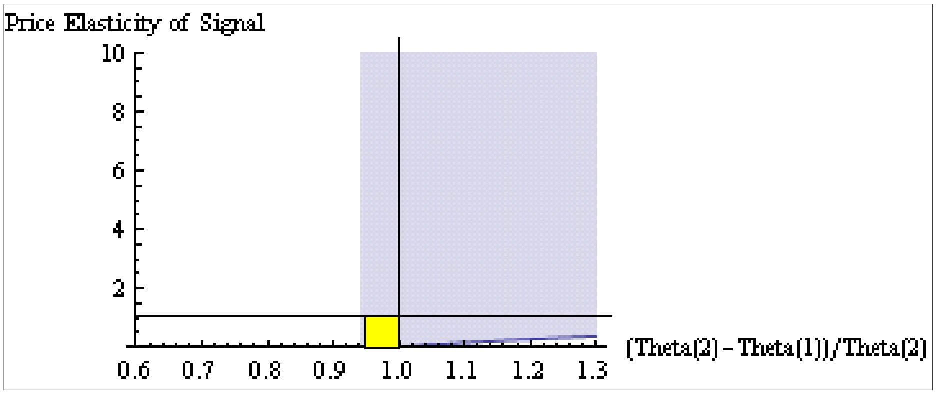

Figure 3

In the above graph, the blue region shows the values of the elasticity for which hyperinflations develop. Since we restrict both weights to be positive, only the region of the horizontal axis below 1 is of interest to us. The yellow square shows that for certain values of the weights, hyperinflations can develop for elasticities less than unity (concave-in-the-signal price functions). Hence, convexity of the price function is *not* the driving force behind the development of hyperinflations.

A Special Case of the Price Function (for illustrational purposes)

In this section, we make specific assumptions about the joint, conditional and marginal distributions of the random variables to get a particular form of the price function (for details, see Appendix A). We also make the assumptions:

$$\phi_1 + \phi_2 = 1$$
$$0 < \phi_1 < 1$$
$$0 < \phi_2 < 1$$

Given the distributional assumptions, the resulting price function is:

$$p_t = (148.84)\, M_{t-1} z_t^{2.4}$$

Note that in this case, the signal elasticity of price is 2.4: i.e. the price function is convex in the signal. Given this form of the price function, we can derive the corresponding inflation rate and the optimal monetary policy as a function of the weights.

Appendix B shows the detailed derivation of the optimal monetary policy. The approximation to this policy is:

$$x_t = \frac{99.18\left(\phi_1 - \sqrt{\phi_1}\sqrt{\phi_1 - 0.00258\phi_2(5.76 + 3.36\log[x_{t-1}])}\right)}{\phi_2}$$

Φ_1 and Φ_2 are the weights on the inflation term and the output deviation term in the loss function, respectively. The young people have beliefs such that

$$\phi_1 \sim i.i.d.\ Uniform[0, 1]$$

In this specification, we choose the parameter values of the real shock process as follows:

$$\theta_t = \theta_{t-1} + \epsilon_t \qquad \text{where } \epsilon_t \text{ is } i.i.d.\ U\left(-\tfrac{1}{30};+\tfrac{1}{30}\right)$$

Taking expectations of both sides, we get:

$$E_t(\theta_t) = \theta_{t-1}$$

Hence, the real shock process is assumed to follow a random walk.

This process looks like:

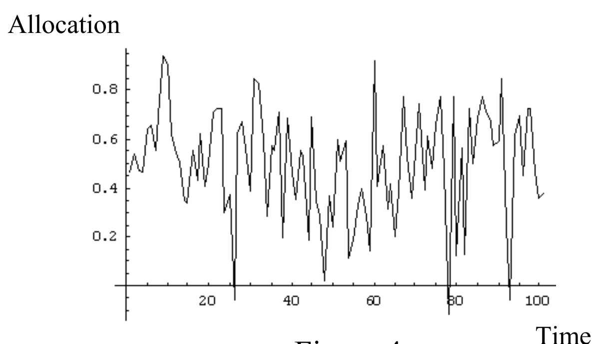

Figure 4

Now we can turn to a graphical examination of the outcomes as a function of the loss function weights.

A. More Weight on the Output Deviation Term

If the output deviation term gets more weight in the loss function, we find that the rates of money creation increase over time, trying to counteract any decreases in the aggregate output. The following graph illustrates this point.

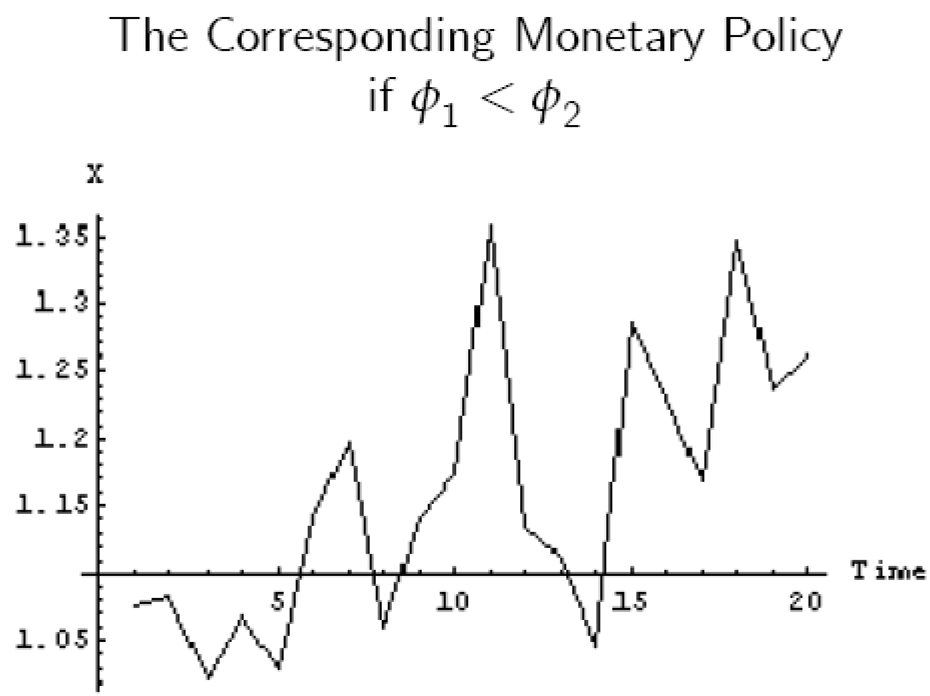

Figure 5

In this case, the increasingly high rates of money creation cause the citizens to observe a sequence of high signals, causing a hyperinflationary outcome (aided by the convexity of the price function in the signal). The corresponding graph is:

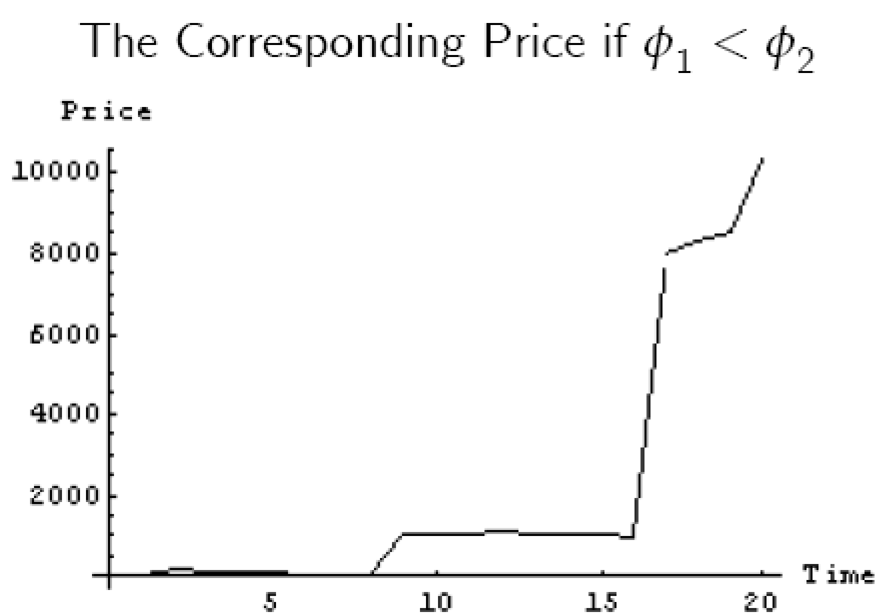

Figure 6

B. **More Weight on the Inflation Term**

If the inflation deviation term gets more weight, rises in the price level cause the government to attempt to counteract this trend by lowering the rate of money creation. As the following graph shows:

A Realization of the Monetary Policy if $\phi_1 > \phi_2$

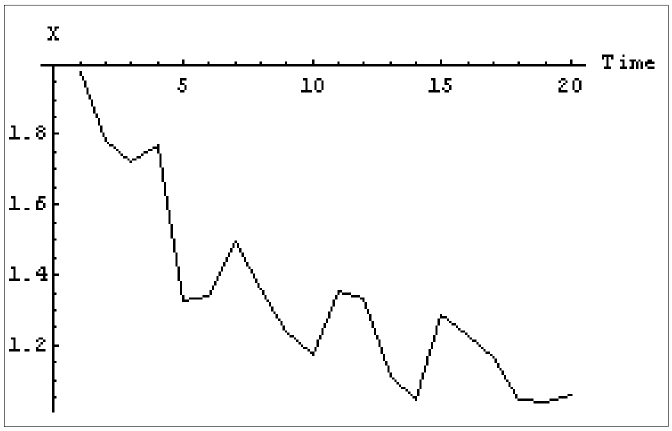

Figure 7

Altogether, a high output term causes the rate of money expansion to increase as the signal increases, while a high inflation term causes the rate of money expansion to decrease as the signal increases. Therefore, the total reaction of the rate of money expansion to changes in the signal, and the resulting inflation rates, depend on the relative weights of the loss function.

Ceteris paribus, the following relationship holds:

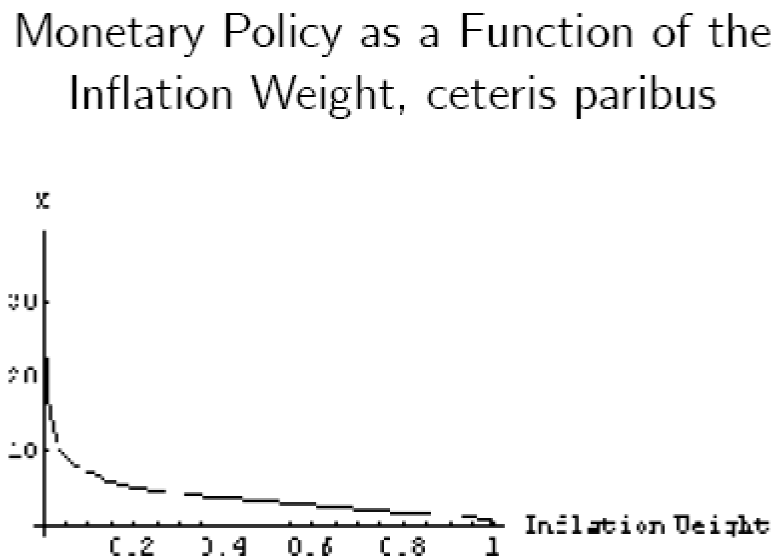

Figure 8

In addition to examining the direction of monetary policy change in response to changes in the signal, it is also interesting to see the magnitude of this reaction. As the following graph shows, a higher inflation term weight causes the monetary policy to be less responsive to changes in the real shock and the signal.

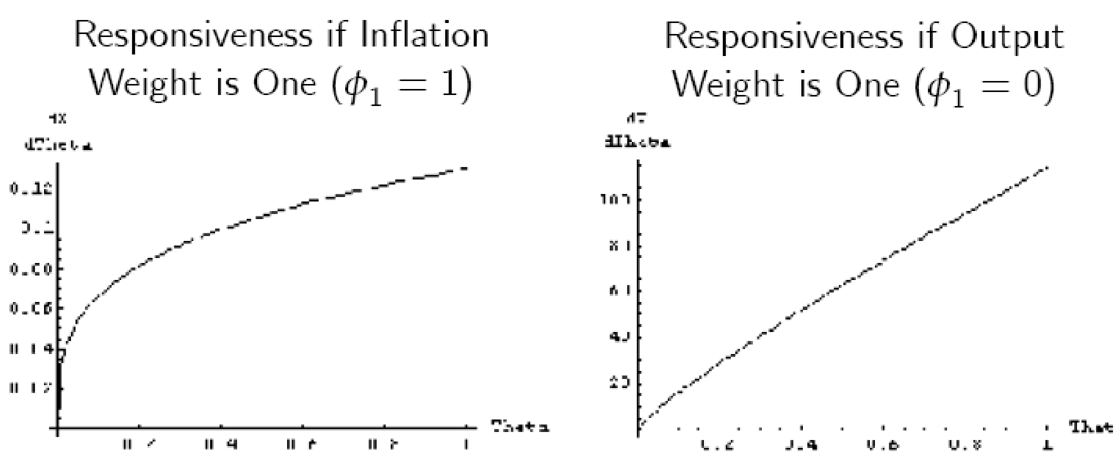

Figure 9

This section has shown that both the direction of change and the responsiveness of the monetary policy depend on the relative weights in the loss function.

Conclusion

We have developed a signal extraction model of "islands" in the spirit of Lucas (1972), whereby monetary policy results in real effects, while consistent with rational expectations and the quantity theory of money. We extended the original "island" model in several ways. First, we developed a multi-period model where the "allocation" variable (the share of the young people on a given island in period t) is realized each period and follows a dynamic stochastic process. Second, we added *endogenous* monetary policy, where the government sets the rate of money creation in order to minimize a loss function over inflation and output deviations (with the weight on the various terms unknown to the young). The signal extraction problem in the model comes from young people's inability to decipher the price function's signal about the loss function weights, and the allocation variable. Hence, monetary policy has real effects.

In the certainty case (where the exact value of the rate of money expansion is announced to the public), the neutrality of money results continue to hold. In the uncertainty case (when the citizens do not know the terms in the loss function), the inflation outcome depends on (1) the signal elasticity of the price function, and (2) the relative weights on the deviations in the loss function. Using Cagan's definition of hyperinflations, we derived which elasticity and loss function weight combinations result in hyperinflations. Interestingly, the price function need not be convex in the signal for hyperinflations to develop.

Appendix A

This Appendix shows the derivation of the price function using the guess-and-verify method. To begin with, assume the random variables are government by the following joint, conditional and marginal distributions:

$$(a) \qquad \theta_t \; Uniform \; (0,1)$$
$$(b) \qquad g(\theta_{t+1} \mid \theta_t) : Uniform \left(\tfrac{1}{2}\eta\theta_{t-1}; \tfrac{3}{2}\eta\theta_{t-1}\right)$$
$$(c) \qquad f(\theta_t, z_t) = \frac{\theta_t + z_t}{\theta_t + 1} Exp[-z_t)]$$

$$(d) \qquad f(\theta_t \mid z_t) = \frac{\theta_t + z_t}{(\theta_t + 1)(1 - \log 2 + z_t \log 2)}$$

$$(e) \qquad p(z_t, z_{t+1}) = \exp[-z_{t+1} - z_t](z_{t+1} - \log 2 + z_t \log 2)$$

$$(f) \qquad p(z_{t+1} \mid z_t) = \frac{\exp[-z_{t+1}](z_{t+1} - \log 2 + z_t \log 2)}{1 - \log 2 + z_t \log 2}$$

Figure 1A: Joint, conditional and marginal distributions of random variables

As shown in the body of the text, the equilibrium money demand function takes the form:

$$\lambda_t = M_{t-1} z_t$$

Substituting this into (3'):

$$(3'') \qquad p_t = \alpha^{-1}(M_{t-1})^{1-\alpha}(z_t)^{1-\alpha} E_t\left[\left(\frac{x_{t+1}}{p_{t+1}}\right)^{\alpha} \mid z_t\right]^{-1} = \frac{(M_{t-1})^{1-\alpha}}{\alpha(z_t)^{\alpha-1} E_t\left[\left(\frac{x_{t+1}}{p_{t+1}}\right)^{\alpha} \mid z_t\right]}$$

Rearranging:

$$(3''') \qquad p_t = \frac{(M_{t-1})^{1-\alpha}}{\alpha(z_t)^{\alpha-1} E_t\left[\left(\frac{x_{t+1}}{p_{t+1}}\right)^{\alpha} \mid z_t\right]}$$

Now, notice that the signal z_t enters the price function p_t via two terms:

First term: $(z_t)^{\alpha-1}$

Second term: $E_t\left[\left(\frac{x_{t+1}}{p_{t+1}}\right)^{\alpha} \mid z_t\right]^{-1}$

Now, we make the conjecture that the price function p_t takes the form:

$$p_t = \frac{M_{t-1} z_t}{\alpha^{\frac{1}{1-\alpha}} \Gamma(z_t)}$$

Where:

z_t = "direct effect" of the signal z_t
$\Gamma(z_t)$ = "indirect effect" through the conditional expectations.

Evaluating this guess at t +1, plugging into (3'') above, and using $X_t = \theta_t^* z_t$, we get:

$$p_t = \frac{(M_{t-1})^{1-\alpha} (M_{t-1})^{\alpha} (z_t)^{1-\alpha} (z_t)^{\alpha}}{\alpha \alpha^{\frac{\alpha}{1-\alpha}} E_t\left[\left(\frac{\theta_{t+1} z_{t+1}^{2(1-\alpha)} \Gamma(z_{t+1})}{\theta_t} \right)^{\alpha} \mid z_t \right]} = \frac{M_{t-1} z_t}{\alpha^{\frac{1}{1-\alpha}} E_t\left[\left(\frac{\theta_{t+1} \Gamma(z_{t+1})}{\theta_t} \right)^{\alpha} \mid z_t \right]}$$

For this guess to hold, we need:

$$\Gamma(z_t) = E_t\left[\left(\frac{\theta_{t+1} \Gamma(z_{t+1})}{\theta_t} \right)^{\alpha} \mid z_t \right]$$

Now, let:

$$\Gamma(z_t) = (z_t)^{w}$$

Then we have:

$$(**) \qquad (z_t)^{w} = E_t\left[\left(\frac{\theta_{t+1}}{\theta_t} (z_{t+1})^{w} \right)^{\alpha} \mid z_t \right]$$

We need to identify the value of *w* so that (**) holds, given the distributions shown in Figure 1A above. We can rewrite the (**) as:

$$E_t\left[\left(\frac{\theta_{t+1}}{\theta_t} \right)^{\alpha} (z_{t+1})^{\alpha w} \mid z_t \right] = \int \left(\frac{\theta_{t+1}}{\theta_t} \right)^{\alpha} (z_{t+1})^{\alpha w} dF(\theta_{t+1} \mid z_{t+1}) dP(z_{t+1} \mid z_t) F(\theta_t \mid z_t)$$

Then (**) becomes:

$$(z_t)^w = \int \left(\frac{\theta_{t+1}}{\theta_t} \right)^{\alpha} (z_{t+1})^{\alpha w} f(\theta_{t+1} \mid z_{t+1}) p(z_{t+1} \mid z_t) f(\theta_t \mid z_t) d\theta_{t+1} d\theta_t dz_{t+1}$$

Using the distributions specified in Figure 1A, this becomes:

$$(z_t)^w = \int_0^{\infty}\int_0^1\int_0^1 \left(\left(\frac{\theta_{t+1}}{\theta_t} \right)^{\alpha} (z_{t+1})^{\alpha w} \right) \left(\frac{\theta_{t+1} + z_{t+1}}{(\theta_{t+1}+1)(1-\log 2 + z_{t+1}\log 2)} \right)$$
$$\left(\frac{\exp[-z_{t+1}](z_{t+1} - \log 2 + z_t \log 2)}{1 - \log 2 + z_t \log 2} \right) \left(\frac{\theta_t + z_t}{(\theta_t + 1)(1 - \log 2 + z_t \log 2)} \right) d\theta_{t+1} d\theta_t dz_{t+1}$$

Evaluating the right-hand side of the above equation and solving for the parameters, we find the unique values as:

$$w = -1.4$$
$$\alpha = 0.19$$

Plugging these parameter values into the price function p_t, we get:

$$(5) \qquad p_t = \frac{M_{t-1} z_t^{1-w}}{\alpha^{\frac{1}{1-\alpha}}}$$

as desired.

Appendix B

This appendix describes the derivation of the optimal rate of money creation X_t. As explained in the body of the text, the goal of the government is to choose the rate of money creation X_t so as to minimize the loss function:

$$Min_{x_t} \qquad \phi_1 \pi_t^2 + \phi_2 (N_t - \bar{N}_t)$$

Where the inflation term Π_t can be expended as:

$$\pi_t = \log p_t - \log p_{t-1} = \log[\frac{M_{t-2} x_t z_t^{2.4}}{0.128}] - \log[\frac{M_{t-2} z_{t-1}^{2.4}}{0.128}] = 3.4 \log[x_t] - 2.4 \log[\theta_t] - 2.4 \log[z_{t-1}]$$

and the aggregate output N_t can be written as:

$$N_t = \theta_t n_t = \theta_t \frac{\lambda_t}{p_t} + \theta_t \alpha^{\frac{1}{1-\alpha}} = \theta_t^{2.4} \frac{0.128}{x_t^{1.4}} + \theta_t \alpha^{\frac{1}{1-\alpha}}$$

Substituting the inflation term and the output term into the loss function, the minimization problem becomes:

$$Min_{x_t} \qquad \phi_1 (3.4 \log[x_t] - 2.4 \log[\theta_t] - 2.4 \log[z_{t-1}])^2 + \phi_2 \left(\theta_t^{2.4} \frac{0.128}{x_t^{1.4}} + \theta_t \alpha^{\frac{1}{1-\alpha}} - \bar{N}_t \right)^2$$

We make the following assumptions:

$$\phi_1 + \phi_2 = 1$$
$$0 < \phi_1 < 1$$
$$0 < \phi_2 < 1$$

Assuming the general regularity conditions hold, we get the first-order condition:

$$6.8\phi_1 (3.4(x_t - 1) - 2.4 \log[\theta_t] - 2.4 \log[z_{t-1}]) - 0.3584\phi_2 \left(\theta_t^{2.4} \frac{0.128}{x_t^{1.4}} + \theta_t \alpha^{\frac{1}{1-\alpha}} - \bar{N}_t \right) \left(\theta_t^{2.4} \frac{1}{x_t^{1.4}} \right) = 0$$

Taking a linear approximation of this first-order condition, and rearranging, the approximate optimal value of the rate of money creation X_t becomes:

$$(9) \qquad x_t = \frac{99.18\left(\phi_1 - \sqrt{\phi_1}\sqrt{\phi_1 - 0.00258\phi_2(5.76 + 3.36 \log[x_{t-1}])}\right)}{\phi_2}$$

as given in the body of the text.

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